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***Abstract*— "Ant algorithms" have been proposed to solve a variety of problems arising in optimization and distributed control. They form a subset of the larger class of "Swarm Intelligence" algorithms. The central idea is that a "swarm" of relatively simple agents can interact through simple mechanisms and collectively solve complex problems. Instances that exemplify the above idea abound in nature. The abilities of ant colonies to collectively accomplish complex tasks have served as sources of inspiration for the design of "Ant algorithms". Examples of "Ant algorithms" are the set of "Ant Routing" algorithms that have been proposed for communication networks. We analyze in this paper Ant Routing Algorithms for packet-switched wireline networks. The algorithm retains most of the salient and attractive features of Ant Routing Algorithms. The scheme is a multiple path probabilistic routing scheme, that is fully adaptive and distributed. Using methods from adaptive algorithms and stochastic approximation, we show that the evolution of the link delay estimates can be closely tracked by a deterministic ODE system. A study of the equilibrium points of the ODE gives us the equilibrium behavior of the routing algorithm, in particular, the equilibrium routing probabilities, and mean delays in the links under equilibrium. We also show that the fixed-point equations that the equilibrium routing probabilities satisfy are actually the necessary and sufficient conditions of an appropriate optimization problem. Simulations supporting the analytical results are provided.**

I. INTRODUCTION

"Ant algorithms" constitute a class of algorithms that have been proposed to solve a variety of problems arising in optimization and distributed control. They form a subset of the larger class of what are referred to as "Swarm Intelligence" algorithms, a topic extensively discussed in the book by Bonabeau, Theraulez, and Dorigo [5]. The central idea here is that a "swarm" of relatively simple agents can interact through simple mechanisms and collectively solve complex problems. Instances that exemplify the above idea abound in nature. Bonabeau, Theraulez, and Dorigo [5] give examples of insect societies like those of ants, honey bees, and wasps, which accomplish fairly complex tasks of building intricate nests, finding food, responding to external threats etc., even though the individual insects themselves have limited capabilities. The abilities of ant colonies to collectively accomplish complex tasks have served as sources of inspiration for the design of "Ant algorithms".

Examples of "Ant algorithms" are the set of "Ant-Based Routing" algorithms that have been proposed for communication networks. It has been observed in an experiment conducted by biologists called the double bridge experiment [7], that a colony of ants, when presented with two paths to a source of food, is able to collectively converge to the shorter path. Every ant lays a trail of a chemical substance called *pheromone* as it walks along a path; subsequent ants follow and reinforce this trail. This leads progressively to a large accumulation of pheromone on the shorter path, which is how ants discover the shorter path. Most of the Ant-Based Routing Algorithms (called Ant Routing Algorithms, for short) proposed in the literature are inspired by this basic idea. These algorithms employ probe packets called ant packets (analogues of ants) to explore the network and measure various quantities related to network routing performance like link and path delays. These measurements are used to construct and update the routing tables at the network nodes. The update algorithms tend to reinforce those outgoing links which lead to paths with lower delays.

Schoonderwoerd *et. al.* [11], [5] tested an Ant Routing Algorithm on the British Telecom telephone network, and reported superior performance compared to other algorithms including shortest paths based schemes. This generated interest in the study of Ant Routing Algorithms for both connection-oriented networks and for packet-switched networks (both wired and wireless). Ant Routing Algorithms for wireless networks have been proposed and analyzed in papers by Baras and Mehta [1] and Gunes *et. al.* [9].

We consider in this paper packet-switched wired networks. Ant Routing Algorithms for such networks have been proposed in the works of Di Caro and Dorigo [8] and Bean and Costa [2]. Though a large number of ant routing algorithms have been proposed, very few analytical studies are available in the literature. Examples of such analytical studies can be found in [12], [10], and [2]. Yoo, La, and Makowski [12] and Gutjahr [10] consider very simple network cases where the delays in the links of the network are deterministic quantities, independent of the offered traffic loads. They show that ant algorithms converge to the shortest path solutions in the network. Bean and Costa [2] propose a multiple-path routing scheme which tries to form estimates of the delays and use them to update routing tables. The link delays can be stochastically varying. The authors employ a time-scale separation approximation whereby the delay estimates are computed before the routing probabilities are updated. They find that results from a numerical study based on their model and those from simulations agree well. However, the exact nature of the time-scale separation is not clear nor is any convergence result provided. Borkar and Kumar [6]

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use stochastic approximation methods to prove convergence of their scheme called Wardrop routing. Their framework is similar to ours - they have a delay estimation scheme and a probability update scheme which utilizes the delay estimates. Their probability update scheme moves on a slower time-scale than their delay estimation scheme. By two time-scale stochastic approximation methods they study convergence of their algorithm.

We consider the algorithm proposed by Bean and Costa [2] in this paper. The algorithm retains most of the attractive features of Ant Routing Algorithms. The scheme is fully adaptive and distributed. We consider in this paper the simple routing scenario where data traffic entering a single source node has to be routed to a single destination node, and there are N available parallel paths between them. We model the arrival processes and packet lengths of both the ant and the data streams that arrive at the source node, and argue, using methods from adaptive algorithms and stochastic approximation, that the evolution of the link delay estimates can be closely tracked by a deterministic ODE system, when the step size of the estimation scheme is small. Then a study of the equilibrium points of the ODE gives us the equilibrium behavior of the routing algorithm; in particular, the equilibrium routing probabilities, and mean delays in the N paths under equilibrium can be obtained. We also show that the fixed-point equations that the equilibrium routing probabilities satisfy are actually the necessary and sufficient conditions of a convex minimization problem.

Our paper is organized as follows. Section II provides the general framework of ant routing and the routing scheme we consider. Section III provides a detailed description of our N parallel paths model. Section IV contains an analysis of the routing scheme. Section V provides illustrative simulation results, and in Section VI we provide a few conclusive remarks.

II. GENERAL FRAMEWORK OF ANT ROUTING ALGORITHMS AND OUR SCHEME

We provide, in this section, a brief description of the general framework of ant routing for a (wired) communication network. The framework that we follow is the one described in Di Caro and Dorigo [7], [8]. Alongside, we describe the scheme due to Bean and Costa [2], that we analyse in this paper.

Every node i in the network maintains two key data structures - a matrix of routing probabilities, the routing table $\mathcal{R}(i)$, and a matrix of various kinds of statistics, called the local network information table, $\mathcal{L}(i)$. For a particular node i, let $\mathcal{N}(i,k)$ denote the set of neighbors of i through which node i routes packets towards destination k. For a communication network consisting of M nodes, the matrix of routing probabilities, $\mathcal{R}(i)$, has $M-1$ columns, corresponding to the $M-1$ destinations towards which node i could route packets, and $M-1$ rows, corresponding to the maximum number of neighbor nodes through which node i could route packets to a particular destination. The entries of $\mathcal{R}(i)$ are the probabilities ϕ_j^{ik}. ϕ_j^{ik} denotes the probability of routing an incoming packet at node i and bound for destination k via the neighbor $j \in \mathcal{N}(i,k)$. The matrix $\mathcal{L}(i)$ has the same dimensions as $\mathcal{R}(i)$, and its (i,j)-th entry contains various statistics pertaining to the route $(i,j,\ldots,k)$. Examples of such statistics could be mean delay and delay variance estimates of the route $(i,j,\ldots,k)$. These statistics are maintained and updated based on the information the ant packets collect about the route. The matrix $\mathcal{L}(i)$ represents the characteristics of the network that are learned by the nodes through the ant packets, based on which local decision-making, and the updating of the routing table $\mathcal{R}(i)$, are done. The iterative algorithms that are used to update $\mathcal{L}(i)$ and $\mathcal{R}(i)$ will be referred to as the *learning algorithms*.

We now describe the mechanism of operation of ant-based routing algorithms. For ease of exposition, we restrict attention to a particular fixed destination node, and consider the problem of routing from every other node to this node, which we label as D.

Forward ant generation and routing. At certain intervals, forward ant (FA) packets are launched from a node i towards the destination node D to discover low delay paths to it. The FA packets sample walks on the graph representing the communication network, based on the current routing probabilities at the nodes. FA packets share the same queues as data packets and so experience similar delay characteristics as data packets. Every FA packet maintains a stack of data structures containing the IDs of nodes in its path and the per hop delays (or other relevant information) encountered. The per hop delay measurements are obtained through time stamping of the packets as they pass from the various nodes.

Backward ant generation and routing. Upon arrival of an FA at the destination node D, a backward ant (BA) packet is generated. The FA packet transfers its stack to the BA. The BA packet then retraces back to the source the path traversed by the FA packet. BA packets travel back in high priority queues, so as to minimize the possibility of outdated or stale measurements. At each node that the BA packet traverses through, it transfers the information that was gathered by the corresponding FA packet. This information is used to update the matrices $\mathcal{L}$ and $\mathcal{R}$ at the respective nodes. Thus the arrival of the BA packet at the nodes triggers the iterative learning algorithms. Of the various learning algorithms that have been proposed in the literature, we consider the one proposed by Bean and Costa [2]. In the following sections of this paper, we analyse this scheme for the simple problem involving routing of incoming traffic between an origin-destination pair through N parallel paths.

Bean and Costa suggest the following scheme for the learning algorithms. Suppose that an FA packet measures the delay Δ_j^{iD} associated with a walk $(i,j,\ldots,D)$. When the corresponding BA packet arrives at node i the delay information is used to update the estimate of the mean delay X_j^{iD} using the simple exponential estimator

$$X_j^{iD} := X_j^{iD} + \eta(\Delta_j^{iD} - X_j^{iD}), \tag{1}$$

where $\eta > 0$ is a small constant. The mean delay estimates X_m^{iD}, corresponding to the other neighbors m of node i, are

left unchanged.

Simultaneously, the routing probabilities at the nodes are updated using the scheme:

$$\phi_j^{iD} = \frac{\left(\frac{1}{X_j^{iD}}\right)^{\beta}}{\sum_{k \in \mathcal{N}(i,D)} \left(\frac{1}{X_k^{iD}}\right)^{\beta}}, \ j \in \mathcal{N}(i,D), \quad (2)$$

where β is a constant positive integer. β influences the extent to which outgoing links with lower delay estimates are favored compared to the ones with higher delay estimates.

We can interpret the quantity $\frac{1}{X_j^{iD}}$ as analogous to the pheromone content on the outgoing link (i,j). Equation (2) shows that the outgoing link (i,j) is more desirable when X_j^{iD}, the delay in routing through j, is smaller (i.e., when the pheromone content is higher).

III. THE N PARALLEL PATHS MODEL

The model that we consider pertains to the routing scenario where arriving traffic at a single source node S has to be routed to a single destination node D. There are N available parallel paths between the source and the destination node through which the traffic could be routed. The network and its equivalent queueing theoretic model are shown in Figures 1 and 2 respectively. The queues represent the output buffers (which we assume to be infinite) at the source and are associated with the N outgoing links. We assume in our model that the queueing delays dominate the propagation and the packet processing delays in the N branches. These additional delay components can be incorporated into our model with no additional complexity, but to keep the discussion simple, we assume they are negligible. Two traffic streams, an ant and a data stream, arrive at the source node S. At node S, every packet of the combined stream is routed with probabilities $\phi_1, \ldots, \phi_N$ (the *current* values) towards the queues $Q_1, \ldots, Q_N$, respectively. These probabilities are updated dynamically based on running estimates of the means of the delays (waiting times) in the N queues. Samples of the delays in the N queues are collected by the ant packets (these are forward ant packets) as they traverse through the queues. These samples are then used to construct the running estimates of the means of the delays in the N queues. We now describe our model in detail.

We model the arrival processes of ant and data stream packets at the source node S as independent Poisson processes of rates λ_A and λ_D packets/sec, respectively. The lengths of the packets of the combined stream constitute an i.i.d. sequence, which is also statistically independent of the packet arrival processes. The capacity of link i is C_i bits/sec ($i = 1, \ldots, N$). We assume that the length of an ant packet is generally distributed with mean L_A bits, and that the length of a data packet is generally distributed with mean L_D bits. If we denote the service times of an ant and a data packet in queue Q_i by the generic random variables S_i^A and S_i^D, then S_i^A and S_i^D are generally distributed (according to some c.d.f.'s, say G_i^A and G_i^D) with means $E[S_i^A] = \frac{L_A}{C_i}$ and $E[S_i^D] = \frac{L_D}{C_i}$, respectively. The ant stream essentially acts as a probing stream in our system collecting samples of delays while traversing through the queues along with the data packets. Thus, the packets of this stream would be much smaller in size compared to the data packets.

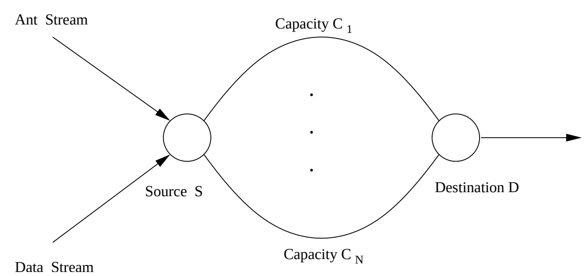

Fig. 1. The network with N parallel paths

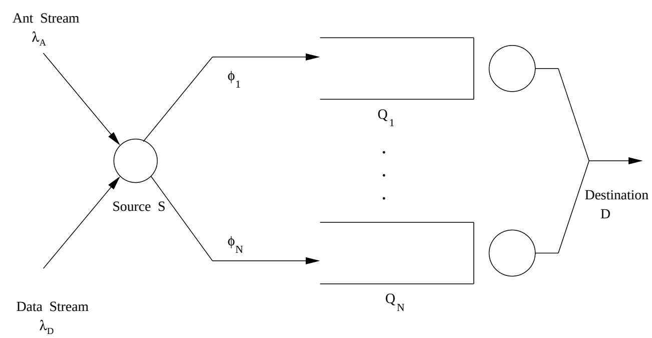

Fig. 2. N parallel paths : The queueing theoretic model

Let $\{\Delta_i(m)\}$ denote the sequence of delays experienced by successive ant packets traversing Q_i. Here delay refers to the total waiting time in the system Q_i (waiting time in the queue plus packet service time). Also, let $\{\delta(n)\}$ denote the sequence of successive arrival times of ant packets at the destination node D. Then the n-th arrival of an ant packet at D occurs at $\delta(n)$. Suppose that this ant packet has arrived via Q_i. We denote the decision variable for routing by $R(n)$; that is, for $i = 1, \ldots, N$, we say that the event $\{R(n) = i\}$ has occurred if the n-th ant packet that arrives at D has been routed via Q_i. $\psi_i(n) = \sum_{k=1}^{n} I_{\{R(k)=i\}}$, thus, gives the number of ant packets that have been routed via Q_i among a total of n ant arrivals at destination D (I_A is the indicator random variable of event A). Once the ant packet arrives, the estimate $X_i(n)$ of the mean of the delay through queue Q_i is immediately updated using a simple exponential averaging estimator

$$X_i(n) = X_i(n-1) + \epsilon \, (\Delta_i(\psi_i(n)) - X_i(n-1)), \quad (3)$$

$0 < \epsilon < 1$, being a small constant.

The delay estimates for the other queues are left unchanged, i.e.,

$$X_j(n) = X_j(n-1), \quad j \in \{1, \ldots, N\}, \ j \neq i. \quad (4)$$

In general, thus, the evolution of the delay estimates in the N queues can be described by the following set of stochastic iterative equations

$$X_i(n) = X_i(n-1) + \epsilon \, I_{\{R(n)=i\}} \Big(\Delta_i(\psi_i(n)) - X_i(n-1)\Big), \; i = 1, \ldots, N, \tag{5}$$

along with a set of initial conditions $X_1(0) = x_1, \ldots, X_N(0) = x_N$.

At time $\delta(n)$, besides the delay estimates, the routing probabilities $\phi_i(n), i = 1, \ldots, N$, are also updated simultaneously according to the equations

$$\phi_i(n) = \frac{(X_i(n))^{-\beta}}{\sum_{j=1}^{N} (X_j(n))^{-\beta}}, \quad i = 1, \ldots, N, \tag{6}$$

β being a constant positive integer. The initial values of the probabilities are $\phi_i(0) = \frac{(x_i)^{-\beta}}{\sum_{j=1}^{N}(x_j)^{-\beta}}, i = 1, \ldots, N$. We note that $\phi_1(n) + \cdots + \phi_N(n) = 1$, for all n.

In the above model we consider only forward ants and do not incorporate the effects of backward ants. More precisely, we assume that the estimates $X_i(.)$ of the means of the delays and the probabilities $\phi_i(.)$ are updated as soon as the (forward) ant packets arrive at the destination D, and this information is available instantaneously thereafter at the source node S. We do not consider thus the additional delay involved as the backward ant packets travel back carrying the delay information to the source. Because backward ants are expected to travel back to the source through priority queues, the effects of delayed information might not be very significant, except for large-sized networks. On the other hand, incorporating the effect of delays in our model introduces additional asynchrony, making the problem harder.

Another important point to note is that our delay estimation scheme (5) is a constant step size scheme. As is well known in the literature on adaptive algorithms (see, for example [3]), this enables the scheme to adapt to (track) long term changes in statistics of the delay processes. This is important for communication networks, because the statistics of arrival processes at the nodes as well as the network characteristics typically change with time.

IV. ANALYSIS OF THE ALGORITHM

We view the routing algorithm, consisting of equations (5) and (6), as a set of discrete stochastic iterations of the type usually considered in the literature on stochastic approximation methods [3]. We provide below the main convergence result which states that, when ϵ is small enough, the discrete iterations are closely tracked by a system of Ordinary Differential Equations (ODEs). In the Appendix, we provide a heuristic analysis which enables us to arrive at the appropriate ODE.

A. The ODE Approximation

An analysis of the dynamics of the system, as given by equations (5) and (6), is fairly complicated. However, when $\epsilon > 0$ is small, a time-scale decomposition simplifies matters considerably. The key observation is that, when ϵ is small, the delay estimates X_i evolve much more slowly compared to the waiting time (delay) processes Δ_i. Also, because the probabilities ϕ_i are (memoryless) functions of the delay estimates X_i, they, too, evolve at the same time-scale as the delay estimates. Consequently, with the vector $(X_1(n), \ldots, X_N(n))$ fixed at $(z_1, ..., z_N)$ (equivalently, $\phi_i(n), i = 1, \ldots, N$, fixed at $\phi_i = \frac{(z_i)^{-\beta}}{\sum_{j=1}^{N}(z_j)^{-\beta}}$, $i = 1, \; \ldots, \; N$), the delay processes $\Delta_i(.), i = 1, \; \ldots, \; N$, can be considered as converged to a stationary distribution, which depends on $(z_1, \ldots, z_N)$. Also, when ϵ is small, the evolution of the delay estimates can be tracked by a system of ODEs. A heuristic analysis of the algorithm, as provided in the Appendix, shows that the ODE system for our case is given by

$$\begin{aligned}
\frac{dz_1(t)}{dt} &= \frac{(z_1(t))^{-\beta}\Big(D_1(z_1(t), \ldots, z_N(t)) - z_1(t)\Big)}{\sum\limits_{j=1}^{N} (z_k(t))^{-\beta}}, \\
\vdots \quad & \quad \vdots \\
\frac{dz_N(t)}{dt} &= \frac{(z_N(t))^{-\beta}\Big(D_N(z_1(t), \ldots, z_N(t)) - z_N(t)\Big)}{\sum\limits_{j=1}^{N} (z_k(t))^{-\beta}},
\end{aligned} \tag{7}$$

with the set of initial conditions $z_1(0) = x_1, \ldots, z_N(0) = x_N$. $D_i(z_1, \ldots, z_N), i = 1, \ldots, N$, are the mean waiting times in the queues under stationarity (as seen by arriving ant packets) with the delay estimates considered fixed at $z_1, \ldots, z_N$.

Formally, the ODE approximation result can be stated as follows (see Benveniste, Metivier, and Priouret [3]). For any fixed $\epsilon > 0$ and for $i = 1, \ldots, N$, consider the piecewise constant interpolation of $X_i(n)$ given by the equations : $z_i^\epsilon(t) = X_i(n)$ for $t \in [n\epsilon, (n+1)\epsilon \;)$, $n = 0, 1, 2, \ldots$, with the initial value $z_i^\epsilon(0) = X_i(0)$. Then the processes $\{z_i^\epsilon(t), t \geq 0\}, i = 1, \ldots, N$, converge to the solution of the ODE system (7) in the following sense : as $\epsilon \downarrow 0$, for any $0 \leq T < \infty$,

$$\sup_{0 \leq t \leq T} |z_i^\epsilon(t) - z_i(t)| \xrightarrow{P} 0, \tag{8}$$

where $\xrightarrow{P}$ denotes convergence in probability.

In order to obtain the evolution of the ODE, we need to compute the quantities $D_i(z_1, \ldots, z_N)$, for our queueing system. We recall that $D_i(z_1, \ldots, z_N), i = 1, \ldots, N$, refer to the means of the waiting times as seen by ant packet arrivals to the queues when the delay estimates are considered fixed at $z_1, \ldots, z_N$. Then the routing probabilities to the N queues are $\phi_i = \frac{(z_i)^{-\beta}}{\sum_{j=1}^{N}(z_j)^{-\beta}}, i = 1, \ldots, N$. We now discuss how to compute the quantities $D_i(z_1, \ldots, z_N)$ given our assumptions on the statistics of the arrival processes and on the packet lengths of the arrival streams.

Under such conditions, every incoming arrival at source S from either of the Poisson streams (the ant or the data stream)

is routed (independent of other arrivals) with probability ϕ_i towards queue Q_i. Thus the incoming arrival process in queue Q_i (for each i) is a superposition of two independent Poisson processes with rates $\lambda_A\phi_i$ and $\lambda_D\phi_i$. Consequently, every incoming packet into Q_i is, with probability $\frac{\lambda_A}{\lambda_A+\lambda_D}$, an ant packet, and with probability $\frac{\lambda_D}{\lambda_A+\lambda_D}$, a data packet. Also, under our assumptions on the statistics of the packet lengths of the arrival streams and on the arrival processes, all of the queues evolve as independent $M/G/1$ queues. The cumulative incoming stream into Q_i is Poisson with rate $(\lambda_A+\lambda_D)\phi_i$, and every incoming packet's service time is distributed according to the c.d.f G_i^A with probability $\frac{\lambda_A}{\lambda_A+\lambda_D}$ and according to the c.d.f. G_i^D with probability $\frac{\lambda_D}{\lambda_A+\lambda_D}$. We further assume that the queues are within the stability region of operation given by the inequalities : $(\lambda_A+\lambda_D)\phi_i E[S_i] < 1,\ i=1,\ \ldots,\ N$, where $E[S_i]$, the mean packet service time in Q_i, is given by $E[S_i] = \frac{\lambda_A E[S_i^A]+\lambda_D E[S_i^D]}{\lambda_A+\lambda_D}$. We note that the average waiting time in the system as experienced by successive ant arrivals to queue Q_i, is the same as the average waiting time in Q_i by the PASTA (Poisson Arrivals See Time Averages) property. Thus, using the Pollaczek-Khinchin formula for the average waiting time and assuming that the queues are stable, we finally obtain the expression for $D_i(z_1,\ldots,z_N)$ ($i=1,\ldots,N$):

$$D_i(z_1,\ldots,z_N) = E[S_i] + \frac{(\lambda_A+\lambda_D)\phi_i E[S_i^2]}{2(1-(\lambda_A+\lambda_D)\phi_i E[S_i])}, \quad (9)$$

where $E[S_i]$ and $E[S_i^2]$ are given by $E[S_i] = \frac{\lambda_A E[S_i^A]+\lambda_D E[S_i^D]}{\lambda_A+\lambda_D}$ and $E[S_i^2] = \frac{\lambda_A E[(S_i^A)^2]+\lambda_D E[(S_i^D)^2]}{\lambda_A+\lambda_D}$, and $\phi_i = \frac{(z_i)^{-\beta}}{\sum_{j=1}^N (z_j)^{-\beta}}$.

Once the expressions for $D_i(z_1,\ldots,z_N)$ are available, we can numerically solve the ODE system (7), starting with the initial conditions $z_1(0),\ldots,z_N(0)$. We observe in our simulations that if we start the system with initial conditions such that we are inside the stability region, the system stays within the stability region thereafter.

B. Equilibrium behavior of the routing algorithm

We now obtain the equilibrium points of the ODE system (7) which would, in turn, enable us to obtain the equilibrium routing behavior of the system. In particular, we can obtain the equilibrium routing probabilities and the mean delays in the system under steady state operation of the network. For ϵ small, the steady state values of the estimates of the average waiting times (delays) in the N queues are approximately given by the components of the equilibrium points, z^*, of the ODE system (7). The equilibrium points of the ODE, z^*, must satisfy the set of equations given by

$$\begin{aligned}
\frac{(z_1^*)^{-\beta}}{\sum\limits_{j=1}^{N}(z_j^*)^{-\beta}} \cdot \Big[D_1(z_1^*,\ldots,z_N^*) - z_1^*\Big] &= 0, \\
\vdots \quad & \quad \vdots \\
\frac{(z_N^*)^{-\beta}}{\sum\limits_{j=1}^{N}(z_j^*)^{-\beta}} \cdot \Big[D_N(z_1^*,\ldots,z_N^*) - z_N^*\Big] &= 0. \quad (10)
\end{aligned}$$

The steady state routing probabilities, $\phi_1^*,\ldots,\phi_N^*$, are related to the average delay estimates, $z_1^*,\ldots,z_N^*$, through the equations, $\phi_i^* = \frac{(z_i^*)^{-\beta}}{\sum_{j=1}^N (z_j^*)^{-\beta}},\ i=1,\ldots,N$. Because we have assumed that our queues are in the stable region of operation, the steady state estimates of average delays must be finite, and so z_i^* must be finite for every $i=1,\ldots,N$. Then the steady state routing probabilities, $\phi_i^* = \frac{(z_i^*)^{-\beta}}{\sum_{j=1}^N (z_j^*)^{-\beta}}$, $i=1,\ \ldots,\ N$, are all strictly positive. Equations (10) then reduce to : $z_i^* = D_i(z_1^*,\ldots,z_N^*), i=1,\ldots,N$. We also notice, from equation (9), that for each i, $D_i(z_1^*,\ldots,z_N^*)$ is a function solely of ϕ_i^*, and so, with a slight abuse of notation, we denote it by $D_i(\phi_i^*)$. Then, utilizing the fact that $\phi_i^* = \frac{(z_i^*)^{-\beta}}{\sum_{j=1}^N (z_j^*)^{-\beta}}$, we find that the equilibrium routing probabilities, $\phi_1^*,\ldots,\phi_N^*$, must satisfy the following fixed-point system of equations

$$\begin{aligned}
\phi_1^* &= \frac{(D_1(\phi_1^*))^{-\beta}}{\sum\limits_{j=1}^{N}(D_j(\phi_j^*))^{-\beta}}, \\
\vdots \quad & \quad \vdots \\
\phi_N^* &= \frac{(D_N(\phi_N^*))^{-\beta}}{\sum\limits_{j=1}^{N}(D_j(\phi_j^*))^{-\beta}}. \quad (11)
\end{aligned}$$

Notice that $\phi_1^*+\cdots+\phi_N^* = 1$.

An important point to note is that the steady state probabilities, $\phi_1^*,\ldots,\phi_N^*$, must not only satisfy the above system of equations, but must all be strictly positive and satisfy the following inequalities which are the stability conditions for the system: $(\lambda_A+\lambda_D)\phi_i^* E[S_i] < 1, i=1,\ldots,N$. We now show that the system of equations (11) are actually the necessary and sufficient optimality conditions for an optimization problem involving the minimization of a convex objective function of $(\phi_1,\ldots,\phi_N)$ subject to the above mentioned constraints. We show as a consequence that, if there exists a solution to the set of equations (11) that also satisfies the above mentioned constraints, then such a solution is unique.

Consider the optimization problem

$$\begin{aligned}
&\text{Minimize } F(\phi_1,\ldots,\phi_N) = \textstyle\sum_{i=1}^N \int_0^{\phi_i} x[D_i(x)]^\beta dx, \\
&\text{subject to } \phi_1+\cdots+\phi_N = 1, \\
&0 < \phi_1 < a_1, \\
&\vdots \\
&0 < \phi_N < a_N,
\end{aligned}$$

where $a_i = \frac{1}{(\lambda_A+\lambda_D)E[S_i]}, i=1,\ldots,N$.

Let us denote by C the set defined by the constraints of the above optimization problem – the feasible set. It is easy to see that C is a convex subset of $\mathbb{R}^N$. It is possible that the set C is empty (for a given set of values of λ_A, λ_D, and $E[S_i], i=1,\ldots,N$), which means that there are no feasible

solutions to the above optimization problem in such a case. We assume, in what follows, that there exists at least one feasible solution to the above optimization problem, i.e., C is non-empty.

Before we attempt to solve the optimization problem, we make certain natural assumptions on the delay functions $D_i(x), i = 1, \ldots, N$. We assume that the functions $D_i(x)$ are positive, real-valued, differentiable and monotonically increasing on their domains of definition. This holds true in most cases of interest, because when the routing probability for an outgoing link increases, the amount of traffic flow into that link also increases, resulting in an increase of the delay. The following proposition characterizes the optimal solutions ϕ^* of the above optimization problem.

Proposition 1: Given the above assumptions on the delay functions $D_i(x), i = 1, \ldots, N$, a probability vector ϕ^* is a local minimum of F over C if and only if ϕ^* satisfies the set of fixed-point equations (11). ϕ^* is then also the unique global minimum of F over C.

Proof: The Hessian of F is a diagonal matrix given by

$$\begin{aligned} \nabla^2 F(\phi_1, \ldots, \phi_N) \quad &= \quad \text{diag}\Big([D_i(\phi_i)]^{\beta-1}\{D_i(\phi_i) + \\ &\qquad \beta\phi_i D_i'(\phi_i)\}\Big), \end{aligned} \tag{12}$$

where $D_i'(.)$ denotes the derivative of $D_i(.)$. Under the above assumptions on the $D_i(x)$'s, $\nabla^2 F(\phi_1, \ldots, \phi_N)$ is positive definite over C, and so F is a strictly convex function on C. Consequently, any local minimum of F is also a global minimum of F over C; furthermore, there is atmost one such global minimum [4].

If $\phi^* = (\phi_1^*, \ldots, \phi_N^*)$ is a local minimum of F over C, we must have (Proposition $2.1.2$ of Bertsekas [4]),

$$\sum_{i=1}^{N} \frac{\partial F}{\partial \phi_i}(\phi^*)(\phi_i - \phi_i^*) \geq 0, \forall \phi \in C. \tag{13}$$

Let us fix a pair of indices $i, j, i \neq j$. Then choose $\phi_i = \phi_i^* + \delta$ and $\phi_j = \phi_j^* - \delta$, and let $\phi_k = \phi_k^*, \forall k \neq i, j$. Now, choosing $\delta > 0$ small enough that the vector $\phi = (\phi_1, \ldots, \phi_N)$ is also in C, the above condition becomes

$$\Big(\frac{\partial F}{\partial \phi_i}(\phi^*) - \frac{\partial F}{\partial \phi_j}(\phi^*)\Big)\delta \geq 0,$$

$$\text{or}, \quad \phi_i^*[D_i(\phi_i^*)]^\beta \geq \phi_j^*[D_j(\phi_j^*)]^\beta.$$

By a similar argument, we can show that $\phi_j^*[D_j(\phi_j^*)]^\beta \geq \phi_i^*[D_i(\phi_i^*)]^\beta$. Thus, the necessary conditions for ϕ^* to be a local minimum are

$$\phi_1^*[D_1(\phi_1^*)]^\beta = \cdots = \phi_N^*[D_N(\phi_N^*)]^\beta.$$

Combining this with the normalization condition, $\phi_1^* + \cdots + \phi_N^* = 1$, gives us the system of equations (11).

The necessary conditions above can also be written in the form

$$\frac{\partial F}{\partial \phi_1}(\phi^*) = \cdots = \frac{\partial F}{\partial \phi_N}(\phi^*).$$

We check that these conditions are also sufficient for ϕ^* to be a local minimum. Suppose $\phi^* \in C$ satisfies the above conditions. Then for every other vector $\phi \in C$, we have $\sum_{i=1}^{N}(\phi_i - \phi_i^*) = 0$. So, the quantity

$$\sum_{i=1}^{N} \frac{\partial F}{\partial \phi_i}(\phi^*)(\phi_i - \phi_i^*) = \frac{\partial F}{\partial \phi_1}(\phi^*) \sum_{i=1}^{N}(\phi_i - \phi_i^*) = 0.$$

Then, because F is convex over C, by Proposition $2.1.2$ of Bertsekas [4], ϕ^* is a local minimum. □

For our model, it is easy to check that the functions $D_i(x)$, as given by (9), are positive, real-valued, differentiable and monotonically increasing on their domains of definition. Thus, there is a unique equilibrium probability vector ϕ^* which satisfies the fixed-point equations (11).

V. SIMULATION RESULTS AND DISCUSSION

We describe in this section an illustrative example. The queueing system as described in Section III has been implemented using a discrete event simulator. We present here results for the case when the number of parallel paths is $N = 3$. The step size ϵ and the parameter β were set at the values 0.002 and 1, respectively.

The ant and data traffic arrival processes are Poisson with rates $\lambda_A = 1$ and $\lambda_D = 1$, respectively. For the ant packets, the service times in the three queues are exponential with means $E[S_1^A] = 1/3.0$, $E[S_2^A] = 1/4.0$ and $E[S_3^A] = 1/5.0$, respectively. For the data packets also, the service times in the three queues are exponential with means $E[S_1^D] = 1/3.0$, $E[S_2^D] = 1/4.0$ and $E[S_3^D] = 1/5.0$, respectively. The initial values of the delay estimates in the three queues were set at $X_1(0) = 0.8$, $X_2(0) = 2.8$, and $X_3(0) = 5.6$. Then, the initial routing probabilities are $\phi_1(0) = 0.7$, $\phi_2(0) = 0.2$, and $\phi_3(0) = 0.1$, which ensures that, initially, we are inside the stability region of the queueing sytem. We observed in our simulations that as the queueing system evolved over time, never did the system become unstable.

Figures 3, 4 and 5 provide plots of the interpolated delay estimates $(z_i^\epsilon(t), i = 1, 2, 3)$ in the three queues, averaged over ten sample paths, versus the ODE approximation, $z_1(t), z_2(t), z_3(t)$, obtained by numerically solving (7). We see that the theoretical ODE tracks the simulated delay estimates fairly well. Figures 6, 7, and 8 provide plots of routing probabilities $\phi_1(n), \phi_2(n)$, and $\phi_3(n)$. The routing probabilities converge to the equilibrium values $\phi_1^* = 3/12, \phi_2^* = 4/12, \phi_3^* = 5/12$, which is actually the unique solution to equations (11). These probabilities are in the reverse order as packet service times in the three queues, with link 3 having the highest equilibrium probability, and link 1 the lowest.

VI. CONCLUSIONS

We have provided convergence results for an Ant Routing Algorithm for a simple network consisting of N parallel paths between a source-destination pair. We have explicitly modeled the link delays using a stochastic queueing model, and we have studied a routing scheme where the routing probabilities are updated based on estimates of path delays. We have also shown that the equilibrium routing probabilities are solutions of a fixed-point system of equations, which in

turn, form the necessary and sufficient optimality conditions for a convex optimization problem. We aim to extend the analysis to the network case, where multiple traffic streams with different destinations share a network of links.

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APPENDIX

The heuristic analysis provided below is largely inspired by Benveniste, Metivier, and Priouret (Section 2.2, Chapter 2) [3]. Let us consider the mean delay estimation scheme given by equation (5). We can write, for a positive integer M,

$$\begin{aligned} X_1(n+M) &= X_1(n) + \epsilon \sum_{k=1}^{M} I_{\{R(n+k)=1\}} \\ &\quad \Big(\Delta_1(\psi_1(n+k)) - X_1(n+k-1)\Big), \\ &\vdots \qquad \vdots \\ X_N(n+M) &= X_N(n) + \epsilon \sum_{k=1}^{M} I_{\{R(n+k)=N\}} \\ &\quad \Big(\Delta_N(\psi_N(n+k)) - X_N(n+k-1)\Big). \end{aligned} \tag{14}$$

If $\epsilon > 0$ is small enough, the vector $(X_1(n), \ldots, X_N(n))$ can be assumed to have not changed much in the discrete interval $\{n, n+1, \ldots, n+M\}$, and we can write the following approximate equations

$$\begin{aligned} X_1(n+M) &\approx X_1(n) + M\epsilon\Big(P_1(n,M) - \\ &\quad Q_1(n,M)X_1(n)\Big), \\ &\vdots \qquad \vdots \\ X_N(n+M) &\approx X_N(n) + M\epsilon\Big(P_N(n,M) - \\ &\quad Q_N(n,M)X_N(n)\Big). \end{aligned} \tag{15}$$

Now we try to find approximations to the quantities $P_i(n,M) = \frac{\sum_{k=1}^{M} I_{\{R(n+k)=i\}}\Delta_i(\psi_i(n+k))}{M}$ and $Q_i(n,M) = \frac{\sum_{k=1}^{M} I_{\{R(n+k)=i\}}}{M}$ for $i = 1, \ldots, N$, when the values of the mean delay estimates $X_1(.), \ldots, X_N(.)$ are considered fixed at $X_1(n), \ldots, X_N(n)$, and M is large. Then the routing probability vector $(\phi_1(.), \ldots, \phi_N(.))$, too, can be regarded as essentially constant in the interval $\{n, \ldots, n+M\}$, because the probabilities are continuous functions of the mean delay estimates. The routing probabilities are then approximately equal to $\phi_i(n) = \frac{(X_i(n))^{-\beta}}{\sum_{j=1}^{N}(X_j(n))^{-\beta}}, i = 1, \ldots, N$. Now, assuming that M is large enough that a law of large numbers effect takes over, the average $\frac{\sum_{k=1}^{M} I_{\{R(n+k)=i\}}}{M}$, which is the fraction of ant packets that have arrived at destination via Q_i when the routing probabilities are $\phi_i(n)$, can be approximated by $\phi_i(n)$. With the routing probabilities fixed, the delay processes $\Delta_i(.)$, can converge to a stationary distribution, the mean under stationarity being denoted by $D_i(X_1(n), \ldots, X_N(n))$. The quantities $\frac{\sum_{k=1}^{M} I_{\{R(n+k)=i\}}\Delta_i(\psi_i(n+k))}{M}$ can then be approximated by $\phi_i(n).D_i(X_1(n), \ldots, X_N(n))$. Note that $D_i(X_1(n), \ldots, X_N(n))$ are the mean waiting times as seen by ant packets. Employing the approximations as described above, we notice from (15) that the evolution of the vector $(X_1(n), \ldots, X_N(n))$ resembles that of a discrete-time approximation to the following ODE system when ϵ is small enough,

$$\begin{aligned} \frac{dz_1(t)}{dt} &= \frac{(z_1(t))^{-\beta}\Big(D_1(z_1(t), \ldots, z_N(t)) - z_1(t)\Big)}{\sum_{j=1}^{N}(z_k(t))^{-\beta}}, \\ &\vdots \qquad \vdots \\ \frac{dz_N(t)}{dt} &= \frac{(z_N(t))^{-\beta}\Big(D_N(z_1(t), \ldots, z_N(t)) - z_N(t)\Big)}{\sum_{j=1}^{N}(z_k(t))^{-\beta}}, \end{aligned} \tag{16}$$

with the set of initial conditions $z_1(0) = x_1, \ldots, z_N(0) = x_N$.

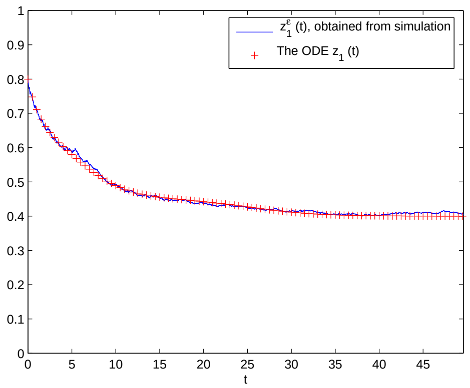

Fig. 3. The ODE approximation for $X_1(n)$

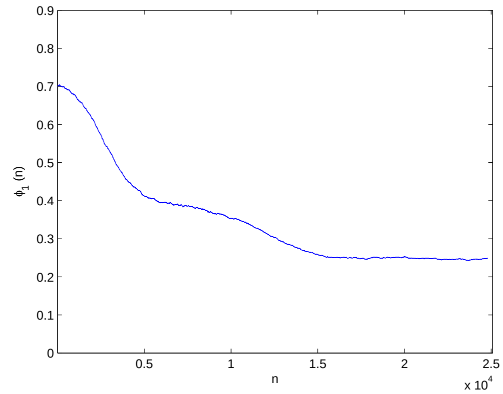

Fig. 6. The plot for $\phi_1(n)$

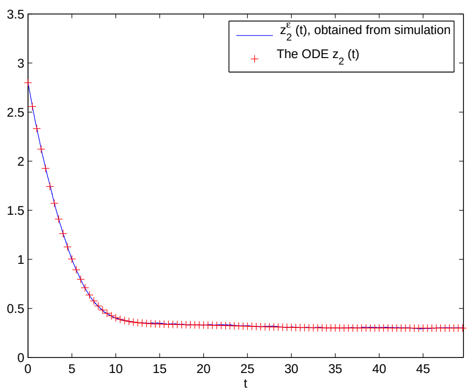

Fig. 4. The ODE approximation for $X_2(n)$

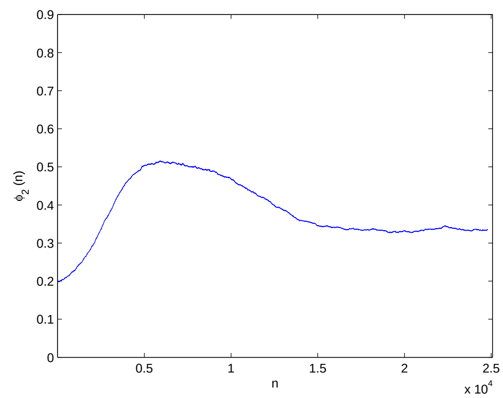

Fig. 7. The plot for $\phi_2(n)$

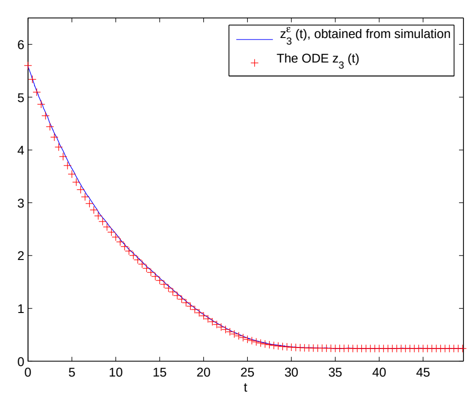

Fig. 5. The ODE approximation for $X_3(n)$

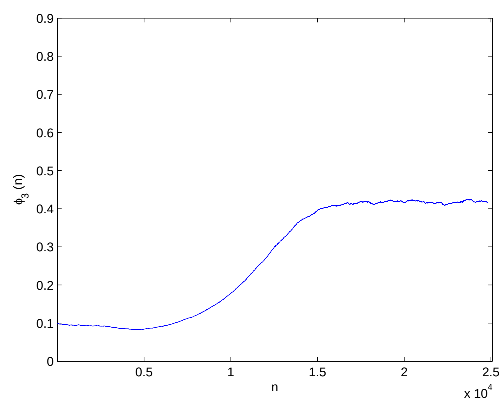

Fig. 8. The plot for $\phi_3(n)$